Thomas Algorithm for Tridiagonal Matrix

Special Matrices

Some matrices have a particular structure that can be exploited to develop efficient solution schemes. Two of those such systems are banded and symmetric matrices.

A banded matrix is a square matrix that has all elements equal to zero, with the exception of a band centered on the main diagonal. Banded systems are frequently encountered in the solution of differential equations. In addition, other numerical methods such as cubic splines involve the solution of banded systems.

The dimensions of a banded system can be quantified by two parameters: the bandwidth BW and the half-bandwidth HBW (Fig. 11.1). These two values are related by $BW = 2HBW + 1$. In general, then, a banded system is one for which $a_{ij} = 0$ if $|i - j| > HBW$. From the book by Chapra:

Efficient elimination methods are described for both.

Heat Transfer Problem

As an example to illustrate the solution mechanism for banded matrices, specifically for tridiagonal matrices, we are going to use a simple problem from the heat transfer field. Assume that the extremes of a cylindrical metal rod are maintained at different temperatures, $0^\circ C$ (left-hand extreme) and $1^\circ C$ (right-hand extreme) and we wish to determine the steady state temperature at interior nodes. To do this, we divide the rod into 11 small imaginary cylinders, the temperatures of each cylinder is represented by the geometric center, named interior nodes as shown in the figure below:
At steady state, the heat transfer equation takes the form:

$$\frac{\partial^2 u}{\partial x^2} = 0 \quad 0 \leq x \leq 1$$

**BC’s:**

$$u = 0 \quad x = 0$$

$$u = 1 \quad x = 1$$

where $u$ is the temperature, and $x$ is the spatial one dimensional coordinate.

Expand the second derivative with central difference representation:

$$\frac{u(i + 1) - 2u(i) + u(i - 1)}{\Delta x^2} = 0 \quad i = 2, 3, 4, \ldots ix - 1$$

Rearrange and simplify

$$u(i - 1) - 2u(i) + u(i + 1) = 0 \quad i = 2, 3, 4, \ldots ix - 1$$

Chose the number of points to be solved as $ix = 11$ (i.e., the number of nodes) and expand from $i = 1, 2, 3, \ldots ix$

$$u(1) - 2u(2) + u(3) = 0$$

$$u(2) - 2u(3) + u(4) = 0$$

$$u(3) - 2u(4) + u(5) = 0$$

$$u(4) - 2u(5) + u(6) = 0$$

$$u(5) - 2u(6) + u(7) = 0$$

$$u(6) - 2u(7) + u(8) = 0$$

$$u(7) - 2u(8) + u(9) = 0$$

$$u(8) - 2u(9) + u(10) = 0$$

$$u(9) - 2u(10) + u(11) = 0$$
Name the coefficients of these equations p, q, r and the right-hand side vector s:

\[
\begin{align*}
    p(1)u(1) + q(1)u(2) + r(1)u(3) &= s(1) \\
    p(2)u(2) + q(2)u(3) + r(2)u(4) &= s(2) \\
    p(3)u(3) + q(3)u(4) + r(3)u(5) &= s(3) \\
    p(4)u(4) + q(4)u(5) + r(4)u(6) &= s(4) \\
    p(5)u(5) + q(5)u(6) + r(5)u(7) &= s(5) \\
    p(6)u(6) + q(6)u(7) + r(6)u(8) &= s(6) \\
    p(7)u(7) + q(7)u(8) + r(7)u(9) &= s(7) \\
    p(8)u(8) + q(8)u(9) + r(8)u(10) &= s(8) \\
    p(9)u(9) + q(9)u(10) + r(9)u(11) &= s(9)
\end{align*}
\]

The matrix form yields a “quasi” tridiagonal matrix:

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Thomas algorithm consists of two steps, direct sweep and inverse sweep.
DIRECT SWEEP

First step

Eliminate \( u(1) \) and \( u(11) \) from the matrix of coefficients in the first and last equations, by passing them to the other side of the equal sign:

For the first equation:

\[
p(1)u(1) + q(1)u(2) + r(1)u(3) = s(1)
\]

\[
q(1)u(2) + r(1)u(3) = s(1) - p(1)u(1)
\]

and for the last equation:

\[
p(9)u(9) + q(9)u(10) + r(9)u(11) = s(9)
\]

\[
p(9)u(9) + q(9)u(10) = s(9) - r(9)u(11)
\]
which square and simplifies the original matrix to a tridiagonal matrix. A tridiagonal system—that is, one with a bandwidth of 3—can be expressed generally as

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Rename:  

\[ s(1) = s(1) - p(1)u(1) \]
\[ s(9) = s(9) - r(9)u(11) \]

The first equation:

\[ q(1)u(2) + r(1)u(3) = s(1) \]  

Eq-1

Normalize the first equation, dividing it by \( q_1 \) (pivot element)

\[ u_2 + \frac{r_1}{q_1} u_3 = \frac{s_1}{q_1} \]

rename: \[ u_2 + \hat{r}_1 u_3 = \hat{s}_1 \]  

Eq-2, where \[ \hat{r}_1 = \frac{r_1}{q_1} \] and \[ \hat{s}_1 = \frac{s_1}{q_1} \]  

Eq-3

Multiplied by \( p_2 \) the first equation in the matrix and subtract it from the second equation:

\[ p_2u_2 + q_2u_3 + r_2u_4 = s_2 \]  

Eq-4

\[ -(p_2u_2 + p_2 \hat{r}_1 u_3) = p_2\hat{s}_1 \]  

Multiply Eq-2 by \( p_2 \)  

Eq-5

\[ (q_2 - p_2\hat{r}_1)u_3 + r_2u_4 = s_2 - p_2\hat{s}_1 \]  

Eq-6

Normalize (6) dividing by \( q_2 - p_2\hat{r}_1 \)

\[ u_3 + \frac{r_2}{q_2 - p_2\hat{r}_1} u_4 = \frac{s_2 - p_2\hat{s}_1}{q_2 - p_2\hat{r}_1} \]  

Eq-7

Rename Eq-7:

\[ u_3 + \hat{r}_2 u_4 = \hat{s}_2 \]  

Eq-8, where \[ \hat{r}_2 = \frac{r_2}{q_2 - p_2\hat{r}_1} \] and \[ \hat{s}_2 = \frac{s_2 - p_2\hat{s}_1}{q_2 - p_2\hat{r}_1} \]  

Eq-9

We repeat the procedure with the 3rd equation in the coefficient matrix and eliminate the \( u_3 \) with the previous equation (Eq-8) in the coefficient matrix:
\( p^3 u + q^3 u + r^3 u = s^3 \) \hspace{2cm} Eq-10

\[-(p^3 r + p^3 r^2 u) = p^3 s^2 \] \hspace{2cm} Eq-11

\[ (q^3 - p^3 r^2) u + r^3 u = s^3 - p^3 s^2 \] \hspace{2cm} Eq-12

Normalize Eq-12 dividing by \( q^3 - p^3 r^2 \)

\[ u + \frac{r^3}{q^3 - p^3 r^2} u = \frac{s^3 - p^3 s^2}{q^3 - p^3 r^2} \] \hspace{2cm} Eq-13

Rename:

\[ u + r^3 u = s^3 \] \hspace{2cm} Eq-14 where \( r^3 = \frac{r^3}{q^3 - p^3 r^2} \) and \( s^3 = \frac{s^3 - p^3 s^2}{q^3 - p^3 r^2} \) \hspace{2cm} Eqs-15

By looking at Eqs-9 and Eqs-15 we derived the recurrence formulas:

\[ \hat{r}_i = \frac{r_i}{q_i - p_i \hat{r}_{i-1}} \] \hspace{2cm} Eq-16 and

\[ \hat{s}_i = \frac{s_i - p_i \hat{s}_{i-1}}{q_i - p_i \hat{r}_{i-1}} \] \hspace{2cm} Eq-17

For the first equation in the coefficient Matrix where \( p_1 \) is not used in the left hand side, it is enough to set \( p_1 = 0 \) for both (Eq-16 and Eq-17) recurrence formulas to yield an expression for the \( r_1 \) and \( s_1 \):

\[ \hat{r}_1 = \frac{r_1}{q_1} \] \hspace{2cm} Eq-3a and

\[ \hat{s}_1 = \frac{s_1}{q_1} \] \hspace{2cm} Eq-3b

NOTE: This doesn’t mean \( p(1) = 0 \) for the right-hand side vector

INVERSE SWEEP

Write the last equation of the coefficient matrix:

\[ p^9 u + q^9 u = s^9 - r^9 u \] \hspace{2cm} Eq-18

As \( i x = 11 \), the above equation can be expressed also as:

\[ p(i x - 2) u(i x - 2) + q(i x - 2) u(i x - 1) = s(i x - 2) \] \hspace{2cm} Eq-19

Write the second to last equation by looking at Eq.-2 or Eq. 7:

\[ u(i x - 2) + \hat{r}(i x - 3) u(i x - 1) = \hat{s}(i x - 3) \] \hspace{2cm} Eq-20

Multiply Eq-20 by \( p(i x - 2) \) and subtract it from Eq-19:
Last equation \[ p(ix - 2)u(ix - 2) + q(ix - 2)u(ix - 1) = s(ix - 2) \]

Second last \[ -(p(ix - 2)u(ix - 2) + p(ix - 2)\hat{r}(ix - 3)u(ix - 1) = p(ix - 2)\hat{s}(ix - 3)) \]

Normalize Eq-20

\[ u(ix - 1) = \frac{s(ix-2)-p(ix-2)\hat{s}(ix-3)}{q(ix-2)-p(ix-2)\hat{r}(ix-3)} \quad \text{Eq-21} \]

Rename the right-hand side of Eq-21:

\[ \hat{s}(ix - 2) = \frac{s(ix-2)-p(ix-2)\hat{s}(ix-3)}{q(ix-2)-p(ix-2)\hat{r}(ix-3)} \quad \text{Eq-22} \]

Then Eq-21 becomes

\[ u(ix - 1) = \hat{s}(ix - 2) \quad \text{Eq-23} \]

For the current example ix=11, therefore ux(ix-1) = u(10), and Eq-21 can be easily computed:

\[ u(10) = \frac{s(9)-p(9)\hat{s}(8)}{q(9)-p(9)\hat{r}(8)} \quad \text{Eq-24} \]

Now write down the normalized equation backwards.

\[ u(ix - 1) = \hat{s}(ix - 2) \quad \text{Eq-23} \]

Now the following equation going up (second to last equation):

\[ u(ix - 2) + \hat{r}(ix - 3)u(ix - 1) = \hat{s}(ix - 3) \quad \text{Eq-20} \]

Solve by u(ix-2):

\[ u(ix - 2) = \hat{s}(ix - 3) - \hat{r}(ix - 3)u(ix - 1) \]

Generalize

\[ u(i) = \hat{s}(i - 1) - \hat{r}(i - 1)u(i + 1) \]

The last computed u(i) will be, e.g., \[ u(2) = \hat{s}(1) - \hat{r}(1)u(3) \]

The formulas to apply then

For the last equation

\[ u(ix - 1) = \hat{s}(ix - 2) \quad \text{Eq-23} \]
For the next ones going from bottom to top:

\[ u(i) = \hat{s}(i - 1) - \hat{r}(i - 1)u(i + 1) \]

e.g. for the last one:

\[ u(2) = \hat{s}(1) - \hat{r}(1)u(3) \]

Summarizing the equations:

**DIRECT SWEEP**

\[ \hat{r}_i = \frac{r_i}{q_i - p_i \hat{r}_{i-1}} \quad \text{Eq}-16 \]
\[ \hat{s}_i = \frac{s_i - p_is_{i-1}}{q_i - p_i \hat{r}_{i-1}} \quad \text{Eq}-17 \]

For Eq-16, the indices run as \(i=2, 3, 4, \ldots, ix-3\) (note that \(r(9) = 0\) is in the other side of the equation and \(\hat{r}=0\) does not exist). Eq-17 applies for \(i=2, 3, \ldots, ix-2\)

For the first equation in the coefficient Matrix where \(i=1\) and \(p(1)\) is in the other side of the equation, the next two equations apply:

\[ \hat{r}_1 = \frac{r_1}{q_1} \quad \text{Eq}-3a \]
\[ \hat{s}_1 = \frac{s_1}{q_1} \quad \text{Eq}-3b \]

**INVERSE SWEEP**

The formulas to apply then

For the last equation

\[ u(ix - 1) = \hat{s}(ix - 2) \quad \text{Eq}-23 \]

For the next ones up

\[ u(i) = \hat{s}(i - 1) - \hat{r}(i - 1)u(i + 1) \]

e.g. for the last unknown to be computed, i.e., \(i=2\):

\[ u(2) = \hat{s}(1) - \hat{r}(1)u(3) \]
Just keep reading
For the example and for developing MATLAB code

\[ u(i - 1) - 2u(i) + u(i + 1) = 0 \quad i = 2, 3, 4, \ldots \ i x - 1 \]

From values of the variables and coefficients:

\begin{align*}
  u(1), u(2), u(3), \ldots, u(11) & \quad u(i), \ldots u(ix) \\
  p(1), p(2), p(3), \ldots, p(9) & \quad p(i), \ldots p(ix-2) \\
  q(1), q(2), q(3), \ldots, q(9) & \quad q(i), \ldots q(ix-2) \\
  s(1), s(2), s(3), \ldots, s(9) & \quad s(i), \ldots s(ix-2) \\
\end{align*}

Actual values:

\begin{align*}
  u(1) &= 0.0 \quad \text{BC} \\
  u(ix) &= u(11) = 1 \quad \text{BC} \\
  p(i) &= 1.0 \quad i = 1, 2, \ldots, ix-2 \\
  q(i) &= -2.0 \quad i = 1, 2, \ldots, ix-2 \\
  r(i) &= 1.0 \quad i = 1, 2, \ldots, ix-2 \\
\end{align*}

The values of \( s(1) \) and \( s(9) \) were renamed, we go back to the original definition:

\begin{align*}
  s(1) &= s(1) - p(1)u(1) = 0 - (1)(0) = 0 & \text{% Note that } p(1) \text{ is not zero for the right-hand side vector} \\
  s(9) &= s(9) - r(9)u(11) = 0 - (1)(1) = -1 \\
  s(i) &= 0 \quad i = 2, \ldots, ix-3 \quad (\text{i.e., } ix-3 = 8) \\
  \dot{s}(1) &= \frac{r(1)}{q(1)} \quad i = 1 \\
  \dot{s}(i) &= \frac{r(i)}{q(i) - p(i)\dot{s}(i-1)} \quad i = 2, 3, \ldots, ix-3 \\
  \dot{s}(1) &= \frac{s(1)}{q(1)} = \frac{0}{-2} = 0 \\
  \dot{s}(i) &= \frac{(s(i) - p(i)\delta(i-1))}{(q(i) - p(i)\dot{s}(i-1))} \quad i = 2, 3, \ldots, ix-2
\end{align*}
**EXACT SOLUTION**

We have chosen a problem that has an exact (and simple) solution, just to show the solution mechanism for the numerical method. The analytical (exact) solution is then

\[
\frac{\partial^2 u}{\partial x^2} = 0 \quad 0 \leq x \leq 1
\]

\[
\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = 0
\]

\[
\frac{\partial u}{\partial x} = K_1 \implies u = K_1 x + K_2
\]

Apply boundary conditions to find the values of \(K_1\) and \(K_2\):

\[
u = 0 \quad x = 0 \implies K_2 = 0
\]

\[
u = 1 \quad x = 1 \implies K_1 = 1
\]

Solution

\[u = x\]