

# Crank Nicolson Solution to the Heat Equation

ME 448/548 Notes

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## Overview

1. Use finite approximations to  $\partial u / \partial t$  and  $\partial^2 u / \partial x^2$ : same components used in FTCS and BTCS.
2. Average contribution from  $t_k$  and  $t_{k+1}$  in the approximation of  $\partial^2 u / \partial x^2$ .
3. Computational formula is (still) *implicit*: all  $u_i^{k+1}$  must be solved simultaneously. Information from  $u_{i-1}^k$ ,  $u_i^k$ , and  $u_{i+1}^k$  are used at each time step in the computation of  $u_i^{k+1}$ .
4. Solution is not significantly more complex than BTCS.
5. Like BTCS, the Crank-Nicolson method is unconditionally stable for the heat equation.
6. Truncation error is  $\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$ . Time accuracy is better than BTCS or FTCS.

## The $\theta$ Method

Evaluate the diffusion operator  $\partial^2 u / \partial x^2$  at both time steps  $t_{k+1}$  and time step  $t_k$ , and use a weighted average

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} = \alpha\theta \left[ \frac{u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1}}{\Delta x^2} \right] + \alpha(1 - \theta) \left[ \frac{u_{i-1}^k - 2u_i^k + u_{i+1}^k}{\Delta x^2} \right] \quad (1)$$

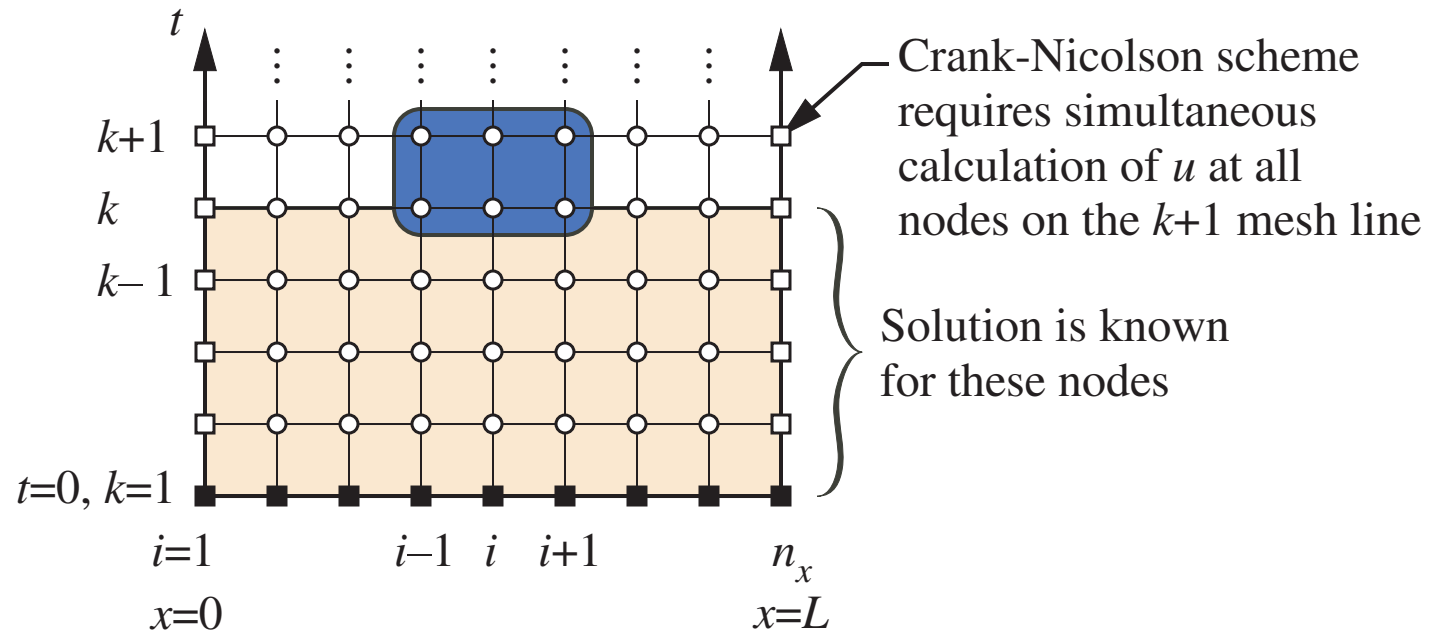
where

$$0 \leq \theta \leq 1$$

$$\theta = 0 \quad \iff \quad \text{FTCS}$$

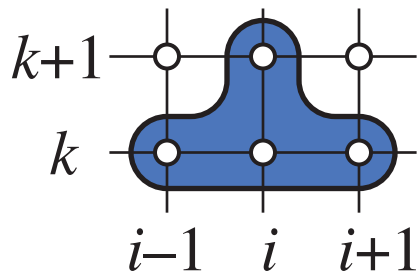
$$\theta = 1 \quad \iff \quad \text{BTCS}$$

## Crank-Nicolson Computational Molecule

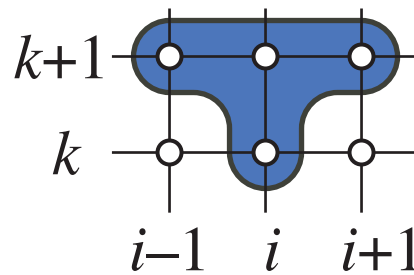


## Compare Computational Molecules

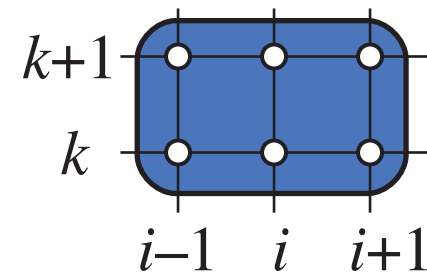
FTCS



BTCS



Crank-Nicolson



FTCS=forward in time, central in space

BTCS=backward in time, central in space

## Crank Nicolson Approximation to the Heat Equation

$$-\frac{\alpha}{2\Delta x^2}u_{i-1}^{k+1} + \left(\frac{1}{\Delta t} + \frac{\alpha}{\Delta x^2}\right)u_i^{k+1} - \frac{\alpha}{2\Delta x^2}u_{i+1}^{k+1} =$$
$$\frac{\alpha}{2\Delta x^2}u_{i-1}^k + \left(\frac{1}{\Delta t} - \frac{\alpha}{\Delta x^2}\right)u_i^k + \frac{\alpha}{2\Delta x^2}u_{i+1}^k \quad (2)$$

## Crank-Nicolson System of Equations

The system of equations has the same structure as BTCS

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 & 0 & 0 \\ c_2 & a_2 & b_2 & 0 & 0 & 0 \\ 0 & c_3 & a_3 & b_3 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & c_{n_x-1} & a_{n_x-1} & b_{n_x-1} \\ 0 & 0 & 0 & 0 & c_{n_x} & a_{n_x} \end{bmatrix} \begin{bmatrix} u_1^{k+1} \\ u_2^{k+1} \\ u_3^{k+1} \\ \vdots \\ u_{n_x-1}^{k+1} \\ u_{n_x}^{k+1} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n_x-1} \\ d_{n_x} \end{bmatrix} \quad (3)$$

where the coefficients of the interior nodes ( $i = 2, 3, \dots, N - 1$ ) are

$$a_i = 1/\Delta t + \alpha/\Delta x^2 = 1/\Delta t - (b_i + c_i),$$

$$b_i = c_i = -\alpha/(2\Delta x^2),$$

$$d_i = -c_i u_{i-1}^k + (1/\Delta t + b_i + c_i) u_i^k - b_i u_{i+1}^k.$$

## demoCN Code

```
% --- Coefficients of the tridiagonal system
b = (-alfa/2/dx^2)*ones(nx,1); % Super diagonal: coefficients of u(i+1)
c = b; % Subdiagonal: coefficients of u(i-1)
a = (1/dt)*ones(nx,1) - (b+c); % Main Diagonal: coefficients of u(i)
at = (1/dt + b + c); % Coefficient of u_i^k on RHS
a(1) = 1; b(1) = 0; % Fix coefficients of boundary nodes
a(end) = 1; c(end) = 0;
[e,f] = tridiagLU(a,b,c); % Save LU factorization

% --- Assign IC and save BC values in ub. IC creates u vector
x = linspace(0,L,nx)'; u = sin(pi*x/L); ub = [0 0];

% --- Loop over time steps
for k=2:nt
    % --- Update RHS for all equations, including those on boundary
    d = - [0; c(2:end-1).*u(1:end-2); 0] ...
        + [ub(1); at(2:end-1).*u(2:end-1); ub(2)] ...
        - [0; b(2:end-1).*u(3:end); 0];
    u = tridiagLUSolve(e,f,b,d); % Solve the system
end
```



## Convergence of FTCS, BTCS and CN

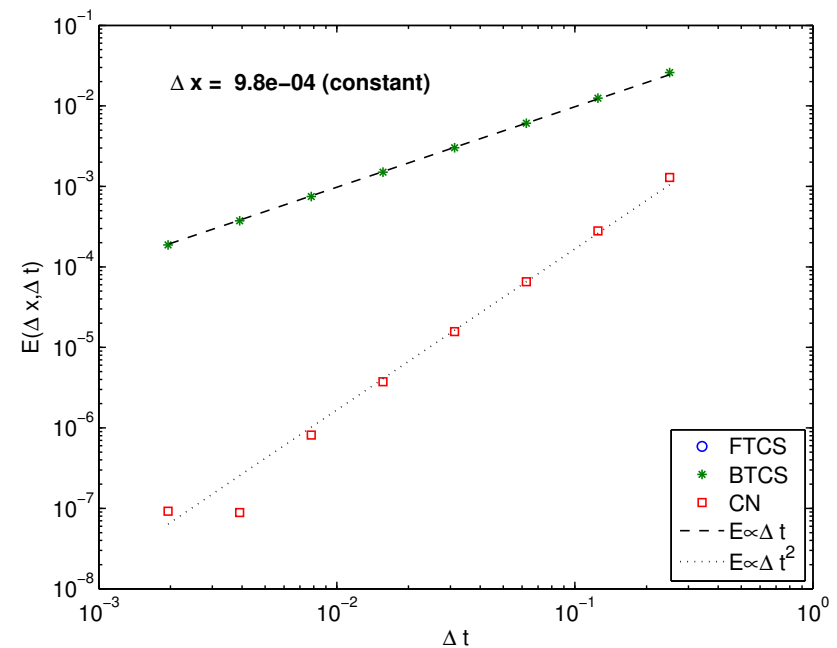
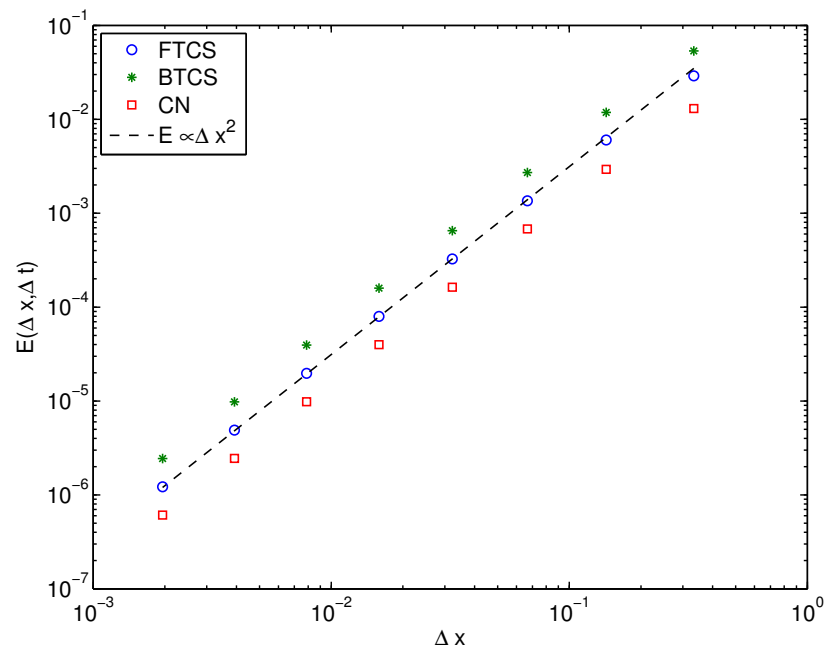
Reduce both dx and dt within the FTCS stability limit

Truncation Errors			Errors				
Scheme	Spatial	Temporal	nx	nt	FTCS	BTCS	CN
			4	5	2.903e-02	5.346e-02	1.304e-02
			8	21	6.028e-03	1.186e-02	2.929e-03
FTCS	$\Delta x^2$	$\Delta t$	16	92	1.356e-03	2.716e-03	6.804e-04
BTCS	$\Delta x^2$	$\Delta t$	32	386	3.262e-04	6.522e-04	1.630e-04
C-N	$\Delta x^2$	$\Delta t^2$	64	1589	7.972e-05	1.594e-04	3.984e-05
			128	6453	1.970e-05	3.939e-05	9.847e-06
			256	26012	4.895e-06	9.790e-06	2.448e-06
			512	104452	1.220e-06	2.440e-06	6.101e-07

Reduce dt while holding dx = 9.775171e-04 (L=1.0, nx=1024) constant

		Errors		
nx	nt	FTCS	BTCS	CN
1024	8	NaN	2.601e-02	1.291e-03
1024	16	NaN	1.246e-02	2.798e-04
1024	32	NaN	6.102e-03	6.534e-05
1024	64	NaN	3.020e-03	1.570e-05
1024	128	NaN	1.502e-03	3.749e-06
1024	256	NaN	7.492e-04	8.154e-07
1024	512	NaN	3.742e-04	8.868e-08
1024	1024	NaN	1.871e-04	9.218e-08

## Convergence of FTCS, BTCS and CN



In the plot of truncation error versus  $\Delta t$  (right hand plot), there is an irregularity at  $\Delta t \sim 3.9 \times 10^{-3}$ . At that level of  $\Delta t$ , and for the chosen  $\Delta x$  (which is constant), the truncation error due to  $\Delta x$  is no longer negligible. Further reductions in  $\Delta t$  alone will not reduce the total truncation error.

## Summary for the Crank-Nicolson Scheme

- The Crank-Nicolson method is more accurate than FTCS or BTCS. Although all three methods have the same spatial truncation error ( $\Delta x^2$ ), the better *temporal* truncation error for the Crank-Nicolson method is big advantage.
- Like BTCS, the Crank-Nicolson scheme is unconditionally stable for the heat equation.
- Like BTCS, a system of equations for the unknown  $u_i^k$  must be solved at each time step. The tridiagonal solver for the 1D heat equation obtains an efficient solution of the system of equations.
- The Crank-Nicolson scheme is recommended over FTCS and BTCS.