BOUNDARY VALUE ODEs

Exercises due on Dec 4, 2014

27.23 Use finite differences to solve the boundary-value ordinary differential equation

$$\frac{d^2u}{dx^2} + 6\frac{du}{dx} - u = 2$$

with boundary conditions $u(0) = 10$ and $u(2) = 1$. Plot the results of u versus x. Use $\Delta x = 0.1$.

27.24 Solve the nondimensionalized ODE using finite difference methods that describe the temperature distribution in a circular rod with internal heat source S

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + S = 0$$

over the range $0 \le r \le 1$, with the boundary conditions

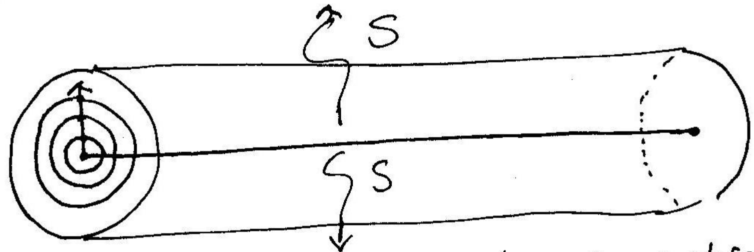
$$T(r = 1) = 1 \qquad \left.\frac{dT}{dr}\right|_{r=0} = 0$$

for $S = 1$, 10, and 20 K/m^2. Plot the temperature versus radius.

Take as an example the problem 27.24 (see also solution manual):

Problem 27.24 BOUNDARY VALUE PROBLEM (1)
"Numerical Methods for Engineers", 6th Edition
By Chapra & Canale

Solve the nondimensionalized ODE that describe the temperature distribution in a a circular rod with internal heat source S



temperature is the same along concentric cylinders
the 1D heat equation in cylindrical coordinates takes the form

$$(1) \quad \frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = -S$$

over the range $0 \le r \le 1$ with the boundary conditions

$$(2) \quad T(r=1) = 1$$

$$(3) \quad \left.\frac{dT}{dr}\right|_{r=0} = 0$$

Solve the equation for three different values of $S = 1, 10, 20$ K/m²
Plot the temperature vs radius

1. Divide the radial coordinate into n finite points (where $i = 1, 2, \ldots n$; i.e. indices of these points start at $i = 1$)

radious

(4) $$\Delta r = \frac{R}{n-1} = \frac{1}{n-1}$$

$i=1 \quad i=2 \quad i=3 \quad \ldots \quad i=n$

2.- The central difference approximation for the derivatives and for the general point i

(5) $$\frac{d^2T}{dr^2} = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta r^2}$$

(6) $$\frac{dT}{dr} = \frac{T_{i+1} - T_{i-1}}{2\Delta r}$$

See Numerical Differentiation class notes (page 8 for 2nd & 1st derivatives of order (h^2)

(7) $$r = \Delta r (i-1)$$

3.- Substituting in the finite difference approximation for the derivatives:

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = -S$$

(8) $$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta r^2} + \frac{1}{\Delta r(i-1)}\left(\frac{T_{i+1} - T_{i-1}}{2\Delta r}\right) = -S$$

Multiplying by Δr^2 the equation

(9) $$T_{i+1} - 2T_i + T_{i-1} + \frac{1}{2(i-1)}T_{i+1} - \frac{1}{2(i-1)}T_{i-1} = -\Delta r^2 S$$

Collecting like terms:

(10) $$\left[1 - \frac{1}{2(i-1)}\right]T_{i-1} + (-2)\,T_i + \left[1 + \frac{1}{2(i-1)}\right]T_{i+1} = -\Delta r^2 S$$

5.- Boundary Condition @ $r=0$ (i.e. $i=1$)

(11) $$\left.\frac{dT}{dr}\right|_{r=0} = 0$$

(12) $$\frac{T_{i+1} - T_{i-1}}{2\Delta r} = 0$$] Central Difference approx $\mathcal{O}(h^2)$

for $i=1$ ← fictitious point

$$\frac{T_2 - T_0}{2\Delta r} = 0 \implies T_2 = T_0 \quad (13)$$

6.- Develop (10) for $i = 1$:

(14) $$\Big[1 - \underbrace{\frac{1}{2(1-1)}}_{-\infty}\Big] T_0 + (-2)T_1 + \Big[1 + \underbrace{\frac{1}{2(1-1)}}_{+\infty}\Big] T_2 = -\Delta r^2 S$$

(15) $$T_0 + (-2)T_1 + T_2 \cancel{+\infty} \cancel{-\infty} = -\Delta r^2 S$$

apply (13) to (15)

$$T_2 + (-2)T_1 + T_2 = -\Delta r^2 S$$

(16) $$(-2)T_1 + 2T_2 = -\Delta r^2 S$$

Alternatively move the first point $i=1$ to $i = 1 + 1/2$ (i.e. $i = 3/2$) collect like terms & cancel some, then move back half a point $i = \frac{3}{2} - \frac{1}{2} = 1$ to end @ the same one.

Or To apply the last boundary condition $T_n = 1$ develop (10) at $i = n-1$

$$\left[1 - \frac{1}{2(n-1-1)}\right] T_{n-1-1} + (-2)\,T_{n-1} + \left[1 + \frac{1}{2(n-1-1)}\right] T_{n-1+1} =$$

$$= -\Delta r^2 S$$

$$\left[1 - \frac{1}{2(n-2)}\right] T_{n-2} + (-2)\,T_{n-1} + \left[1 + \frac{1}{2(n-2)}\right] \cancelto{1}{T_n} = -\Delta r^2 S$$

$$\left[1 - \frac{1}{2(n-2)}\right] T_{n-2} + (-2)\,T_{n-1} = -\Delta r^2 S - \left[1 + \frac{1}{2(n-2)}\right]$$

Resuming

1st Equation

$$(-2)\,T_1 + 2\,T_2 = -\Delta r^2 S$$

Next $i = 2, 3, \ldots n-2$

$$\left[1 - \frac{1}{2(i-1)}\right] T_{i-1} + (-2)\,T_i + \left[1 + \frac{1}{2(i-1)}\right] T_{i+1} = -\Delta r^2 S$$

Last Equation

$$\left[1 - \frac{1}{2(n-2)}\right] T_{n-2} + (-2)\,T_{n-1} = -\Delta r^2 S - \left[1 + \frac{1}{2(n-2)}\right]$$

These system of equations yields a Tridiagonal Matrix which can be easily solved by Thoma's alg.