**Chap 15. The Laplace Equation-Example**

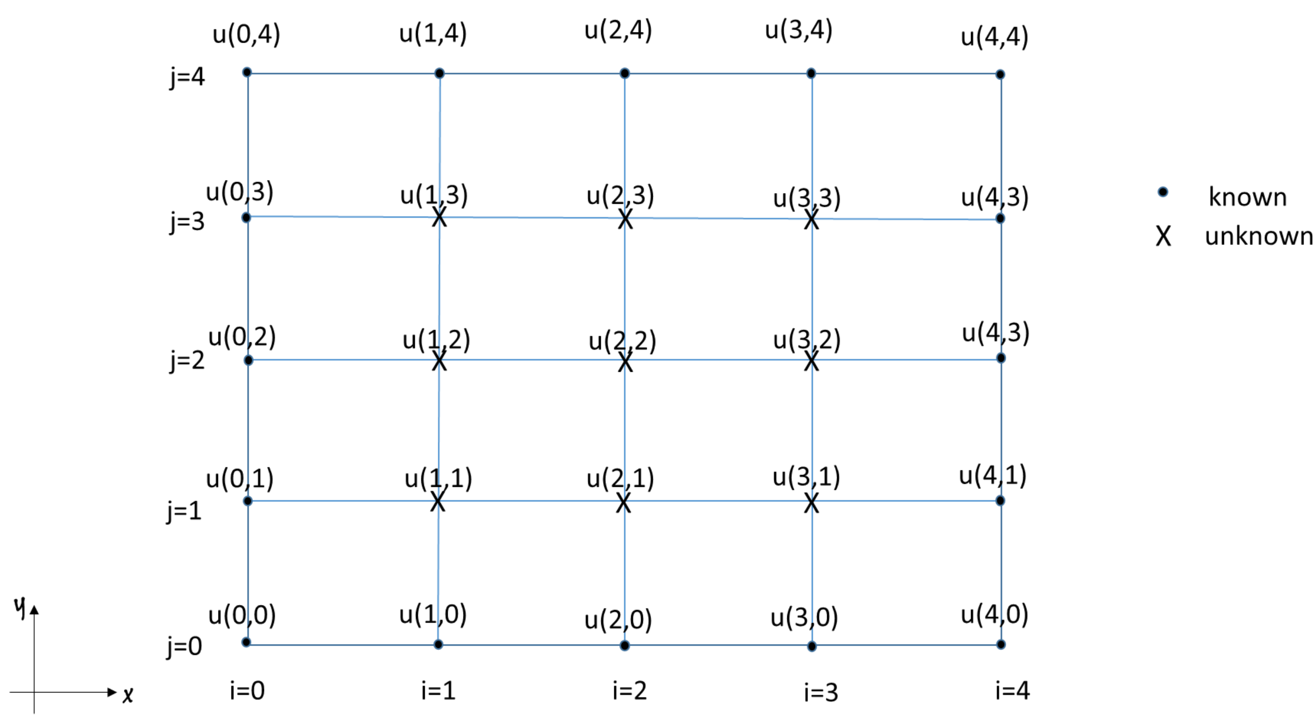
For the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad [1]$$

for $u(x, y)$ defined on $x \in [0, 1], \quad y \in [0, 1]$ with the following four boundary conditions

(I) $u(x, 0) = 3 \quad [2]$
(II) $u(x, 1) = 1 \quad [3]$
(III) $u(0, y) = 3 \quad [4]$
(IV) $u(1, y) = 1 \quad [5]$

(1) Discretize the domain of the problem. For two independent variables use a mesh (also called a grid)—a mesh is a net that is formed by connecting nodes in a predefined manner. Grid points are typically arranged in a rectangular array of nodes. Each axis represents one of the independent variables. Nodes are labeled by indices, i for space and k for time, e.g., i=0,1,2,… ix, and k=0,1,2,…,kx. The quantity Δx or Δy are the discretization parameters, which in particular determine the dimension of the discrete problem. A detailed discretization is shown below. To make it simple, consider just the "toy" discretization/solution below and write the equations for the interior nodes.



(2) Discretization of the PDE by approximating the derivatives with Central Difference Approximation of error $\vartheta(\Delta x^2, \Delta y^2)$ until you find the recurrence formulation.

a. Approximate the 2nd order space derivative with central difference approximation of error order $\vartheta(\Delta x^2)$ at the grid point *(i,k)* using the 'lazy solver' tables:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \quad [6]$$

The space *j* index remains constant as this is a space derivative in x.

b. Approximate the 2nd order space derivative with central difference approximation of error order $\vartheta(\Delta y^2)$ at the grid point *(i,k)* using the 'lazy solver' tables:

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} \quad [7]$$

The space *i* index remains constant as this is a space derivative in y.

Plugging [6] and [7] into [1]

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = 0 \quad [8]$$

At $\Delta x = \Delta y$, this equation can be rearranged as

$$-4u_{i,j} + u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} = 0$$

(3) Discretization of boundary conditions

(I) $u(x,0) = 3$ means $u(x, y = 0) = 3$ means $u(i, j = 0) = 3$ for i=0,1,2,3,4

Yields: u(0,0)=u(1,0)=u(2,0)=u(3,0)=u(4,0)=3

(II) $u(x,1) = 1$

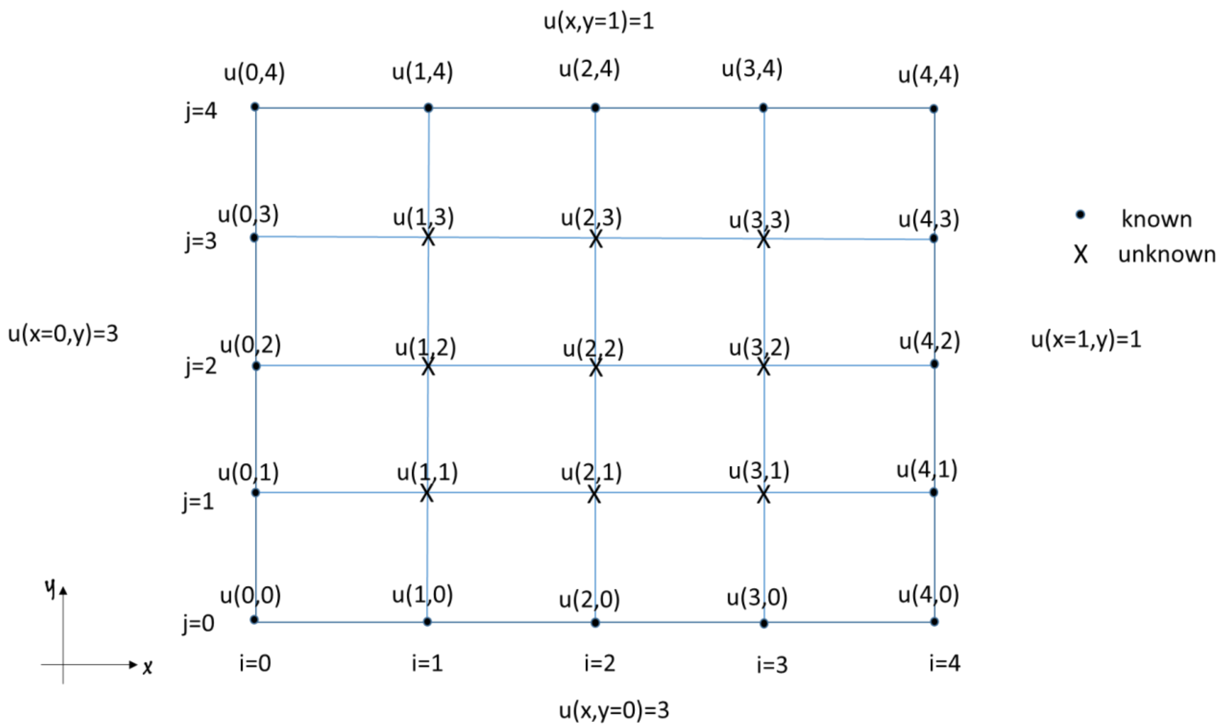
Yields: u(0,4)=u(1,4)=u(2,4)=u(3,4)=u(4,4)=1

(III) $u(0,y) = 3$

Yields: u(0,0)=u(0,1)=u(0,2)=u(0,3)=u(0,4)=3

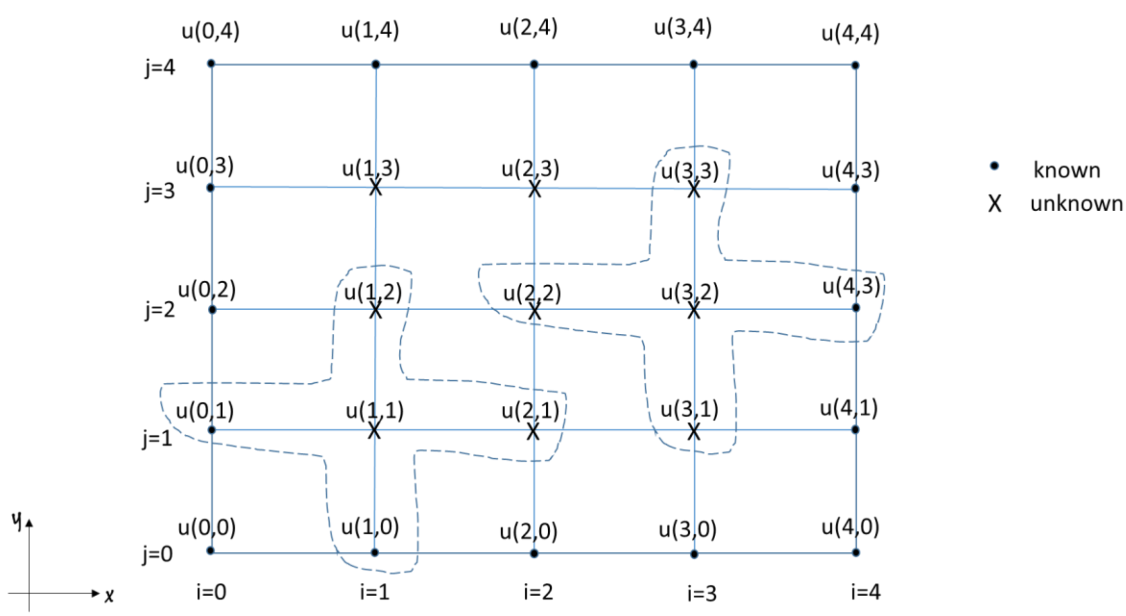
(IV) $u(1,y) = 1$

Yields: u(4,0)=u(4,1)=u(4,2)=u(4,3)=u(4,4)=1




Please notice, that corners values such as u(0,4), u(4,4), u(4,0), and u(0,0) are not used in the discretized equations, i.e., they are not needed for computations.

(4) Express the system of equations to be solved in a matrix form. Before expanding equation (3) for every unknown node, please notice the computational molecule. Here we show two of them:



Now notice there are nine (9) unknowns and we need nine (9) equations to solve the system. The trick to develop the necessary equations is focusing in the values of (i,j) of the interior nodes. For instance, below we show at the left the value of (i,j) combination for interior nodes and to RHS the resulting equation. If the indices of all u(i,j) (known and unknown) are within the grid, you probably are right.

Develop each equation with

node (i=1, j=1):

$$-4u_{1,1} + u_{0,1} + u_{2,,1} + u_{1,0} + u_{1,2} = 0$$

node (i=2, j=1):

$$-4u_{2,1} + u_{1,1} + u_{3,,1} + u_{2,0} + u_{2,2} = 0$$

node (i=3, j=1):

$$-4u_{3,1} + u_{2,1} + u_{4,,1} + u_{3,0} + u_{3,2} = 0$$

node (i=1, j=2):

$$-4u_{1,2} + u_{0,2} + u_{2,,2} + u_{1,1} + u_{1,3} = 0$$

node (i=2, j=2):

$$-4u_{2,2} + u_{1,2} + u_{3,,2} + u_{2,1} + u_{2,3} = 0$$

node( i=3, j=2):

$$-4u_{3,2} + u_{2,2} + u_{4,2} + u_{3,1} + u_{3,3} = 0$$

node (i=1, j=3):

$$-4u_{1,3} + u_{0,3} + u_{2,,3} + u_{1,2} + u_{1,4} = 0$$

node (i=2, j=3):

$$-4u_{2,3} + u_{1,3} + u_{3,,3} + u_{2,2} + u_{2,4} = 0$$

node ( i=3, j=3):

$$-4u_{3,3} + u_{2,3} + u_{4,,3} + u_{3,2} + u_{3,4} = 0$$

The equations reduce to:

node (i=1, j=1):

$$-4u_{1,1} + 3 + u_{2,,1} + 3 + u_{1,2} = 0$$

node (i=2, j=1):

$$-4u_{2,1} + u_{1,1} + u_{3,,1} + 3 + u_{2,2} = 0$$

node (i=3, j=1):

$$-4u_{3,1} + u_{2,1} + 1 + 3 + u_{3,2} = 0$$

node (i=1, j=2):

$$-4u_{1,2} + 3 + u_{2,,2} + u_{1,1} + u_{1,3} = 0$$

node (i=2, j=2):

$$-4u_{2,2} + u_{1,2} + u_{3,,2} + u_{2,1} + u_{2,3} = 0$$

node( i=3, j=2):

$$-4u_{3,2} + u_{2,2} + 1 + u_{3,1} + u_{3,3} = 0$$

node (i=1, j=3):

$$-4u_{1,3} + 3 + u_{2,,3} + u_{1,2} + 1 = 0$$

node (i=2, j=3):

$$-4u_{2,3} + u_{1,3} + u_{3,,3} + u_{2,2} + 1 = 0$$

node ( i=3, j=3):

$$-4u_{3,3} + u_{2,3} + 1 + u_{3,2} + 1 = 0$$

Simplify: Leave unknowns in the LHS and the rest on the other side:

node (i=1, j=1):

$$-4u_{1,1} + u_{2,,1} + u_{1,2} = -6$$

node (i=2, j=1):

$$-4u_{2,1} + u_{1,1} + u_{3,1} + u_{2,2} = -3$$

node (i=3, j=1):

$$-4u_{3,1} + u_{2,1} + u_{3,2} = -4$$

node (i=1, j=2):

$$-4u_{1,2} + u_{2,,2} + u_{1,1} + u_{1,3} = -3$$

node (i=2, j=2):

$$-4u_{2,2} + u_{1,2} + u_{3,,2} + u_{2,1} + u_{2,3} = 0$$

node( i=3, j=2):

$$-4u_{3,2} + u_{2,2} + u_{3,1} + u_{3,3} = -1$$

node (i=1, j=3):

$$-4u_{1,3} + u_{2,,3} + u_{1,2} = -4$$

node (i=2, j=3):

$$-4u_{2,3} + u_{1,3} + u_{3,,3} + u_{2,2} = -1$$

node ( i=3, j=3):

$$-4u_{3,3} + u_{2,3} + u_{3,2} = -2$$

It is a good idea to order the unknowns. In the matrix below we have ordered the unknowns row-by-row of the grid. Another possibility is order them column-by-column of the grid, in that case the coefficient matrix will look different.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| -4 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | | u(1,1) | | -6 |
| 1 | -4 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | | u(1,2) | | -3 |
| 0 | 1 | -4 | 0 | 0 | 1 | 0 | 0 | 0 | | u(1,3) | | -4 |
| 1 | 0 | 0 | -4 | 1 | 0 | 1 | 0 | 0 | | u(2,1) | | -3 |
| 0 | 1 | 0 | 1 | -4 | 1 | 0 | 1 | 0 | | u(2,2) | = | 0 |
| 0 | 0 | 1 | 0 | 1 | -4 | 0 | 0 | 1 | | U(2,3) | | -1 |
| 1 | 0 | 1 | 1 | 0 | 0 | -4 | 0 | 0 | | u(3,1) | | -4 |
| 0 | 0 | 0 | 0 | 1 | 0 | 1 | -4 | 1 | | u(3,2) | | -1 |
| 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | -4 | | u(3,3) | | -2 |

(5) Solve the system of linear equations. The current 9x9 system of equations, can be solved by <u>many</u> methods, one of them is solving the matrix equation by matlab operators. A simple MATLAB script:

```
% LaplaceEquation.m
% Example

clc, clear

A=[-4,1,0,1,0,0,0,0,0;
    1,-4,1,0,1,0,0,0,0;
    0,1,-4,0,0,1,0,0,0;
    1,0,0,-4,1,0,1,0,0;
    0,1,0,1,-4,1,0,1,0;
    0,0,1,0,1,-4,0,0,1;
    1,0,1,1,0,0,-4,0,0;
    0,0,0,0,1,0,1,-4,1;
    0,1,0,0,0,1,0,0,-4];

b=[-6,-3,-4,-3,0,-1,-4,-1,-2]';

x=A\b;
```

OUTPUT

```
>> x
x =
    2.8146
    2.5238
    2.0584
    2.7347
    2.2222
    1.7098
    2.9019
    1.9206
    1.5584
```

The solution above represents only the interior nodes. By adding the constant BCs, the complete solution matrix is obtained.