Numerical Methods PDE-Crank-Nicolson Class Exercise. April-16-2018

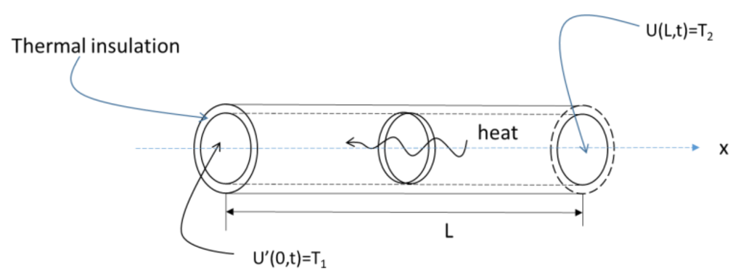
Last name ___________________________ First name _______________ ID ___________________________

PROBLEM#1. CN PDE. Consider the Heat Equation below,

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} \quad [Eq\ 1]$$

Consider the Heat Equation above, a parabolic PDE, as the equation governing the temperature distribution of a thin rod insulated at all points, except at its ends (this means heat exchange only happens at the rod ends) with the following specifications:

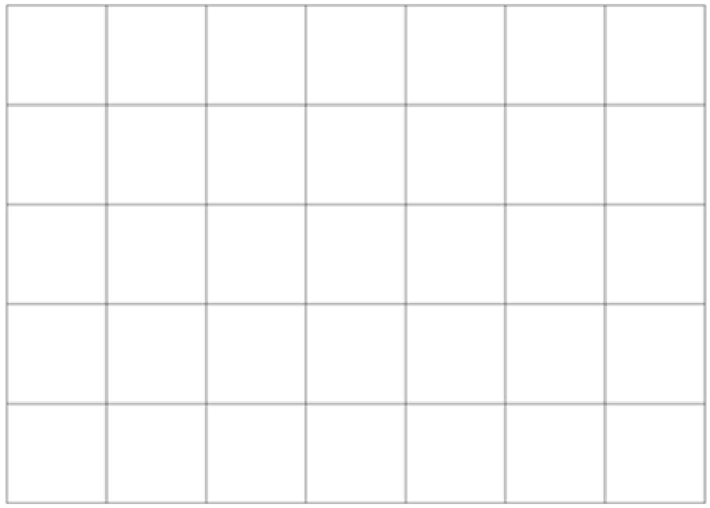
- L = 5 cm (rod length)
- Δx = 1 cm, Δt = 0.1 s, k = 0.835 cm^2/s (thermal diffusivity)
- $\lambda = k\Delta t/\Delta x^2$=0.835(0.1)/$(1)^2$ = 0.0835 (this parameter shows later alongside the discretization of the ODE)
- T_0 = 0 °C
- T_1 = 40 °C
- T_2 = 200 °C



At t = 0, the temperature of interior nodes of the rod is zero and the boundary conditions are

- *IC:* $U(x,0) = T_0$ °C for $0<x<L$
- *BC:* $U(0,t) = T_1$ °C for $t>0$
- *BC:* $U(L,t) = T_2$ °C for $t>0$

(1) **Discretize the domain of the problem.** For two independent variables use a mesh grid, each axis representing one of the axes. Use as running indices, **i** for space and **k** for time, e.g., i=0,1,2,… ix, and k=0,1,2,…,kx. Both starting in zero. Use the grid below:



(2) Discretization of the Differential equation.

Starting with

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} \quad [Eq\ 1]$$

To develop the FTCS scheme, let's handle each term of Eq 1:

a. Approximate the 1st order time derivative with _______ finite difference approximation of $\vartheta(\Delta t^n)$ [n=_____]at the grid point (*i,k+1/2*):

b. Evaluate $\left.\frac{\partial^2 U}{\partial x^2}\right|_{i,k+1/2}$ with ______ finite difference approximation of $\vartheta(\Delta x^n)$ [n=__________]

c. Find the recurrence formulation grouping characteristic parameters, e.g., $\lambda = \frac{\kappa\Delta t}{\Delta x^2}$

(3) Discretize the initial and boundary conditions.

Original Mathematical Expression	Discretized
$U(0,t) = T_1$ for t>0	
$U(L,t) = T_2$ for t>0	
$U(x,T_0) = 0$ for 0<x<L	

(4) Show the CN computational molecule (CM) within the grid of your discretization in question-1.

see GRID above

Answer the following questions:

a) How many unknowns do you have at each time step?____________

b) How many times does the computational molecule fits in the grid during the computation of one time step?______

(5) **System of Equations**. According to your problem discretization and the number of unknowns find the appropriate number of equations to solve for the FIRST time step (i.e. $[t_1, U_i^1]$. We use indexed-variable algebraic equations—Not numbers yet.

(6) Express the problem with numbers for a clean system of linear equations and also in matrix form.

a. System of equations:

b. Matrix Form

(7) Solve either one (6.a) Excel or (6.b) matlab for $0 \leq t \leq 1.0$