Runge-Kutta formulas for solving ordinary differential equations (ODEs) and systems of ODEs.

Model Equation: $\frac{dy}{dt} = f(x,y)$ and initial value $y(t_o) = y_o$

Example: $\frac{dy}{dt} = t + y$ with initial condition $y(t=0) = 1$

Second-order Runge-Kutta:

$$k_1 = \Delta t f(t^n, y^n)$$
$$k_2 = \Delta t f(t^n + \Delta t, y^n + k_1)$$
$$y^{n+1} = y^n + \frac{1}{2}(k_1 + k_2)$$

Fourth-order Runge-Kutta:

$$k_1 = \Delta t f(t^n, y^n)$$
$$k_2 = \Delta t f\left(t^n + \frac{1}{2}\Delta t, y^n + \frac{1}{2}k_1\right)$$
$$k_3 = \Delta t f\left(t^n + \frac{1}{2}\Delta t, y^n + \frac{1}{2}k_2\right)$$
$$k_4 = \Delta t f\left(t^n + \Delta t, y^n + k_3\right)$$
$$y^{n+1} = y^n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

System of ODEs.

Model Equation:

$$\frac{dx}{dt} = f_x(t,x,y); \quad x(t_o) = x_o$$
$$\frac{dy}{dt} = f_y(t,x,y); \quad y(t_o) = y_o$$

Example:

$$\frac{dx}{dt} = xy + t; \quad x(0) = 1$$
$$\frac{dy}{dt} = ty + x; \quad y(0) = -1$$

Fifth-order Runge-Kutta:

$$k_{1,x} = \Delta t f_x\left(t^n, x^n, y^n\right) \qquad k_{1,y} = \Delta t f_y\left(t^n, x^n, y^n\right)$$
$$k_{2,x} = \Delta t f_x\left(t^n + \frac{1}{4}\Delta t, x^n + \frac{1}{4}k_{1,x}, y^n + \frac{1}{4}k_{1,y}\right) \qquad k_{2,y} = \Delta t f_y\left(t^n + \frac{1}{4}\Delta t, x^n + \frac{1}{4}k_{1,x}, y^n + \frac{1}{4}k_{1,y}\right)$$
$$k_{3,x} = \Delta t f_x\left(t^n + \frac{3}{8}\Delta t, x^n + \frac{3}{32}k_{1,x} + \frac{9}{32}k_{2,x}, y^n + \frac{3}{32}k_{1,y} + \frac{9}{32}k_{2,y}\right)$$
$$k_{3,y} = \Delta t f_y\left(t^n + \frac{3}{8}\Delta t, x^n + \frac{3}{32}k_{1,x} + \frac{9}{32}k_{2,x}, y^n + \frac{3}{32}k_{1,y} + \frac{9}{32}k_{2,y}\right)$$
$$k_{4,x} = \Delta t f_x\left(t^n + \frac{12}{13}\Delta t, x^n + \frac{1932}{2197}k_{1,x} - \frac{7200}{2197}k_{2,x} + \frac{7296}{2197}k_{3,x}, y^n + \frac{1932}{2197}k_{1,y} - \frac{7200}{2197}k_{2,y} + \frac{7296}{2197}k_{3,y}\right)$$
$$k_{4,y} = \Delta t f_y\left(t^n + \frac{12}{13}\Delta t, x^n + \frac{1932}{2197}k_{1,x} - \frac{7200}{2197}k_{2,x} + \frac{7296}{2197}k_{3,x}, y^n + \frac{1932}{2197}k_{1,y} - \frac{7200}{2197}k_{2,y} + \frac{7296}{2197}k_{3,y}\right)$$
$$k_{5,x} = \Delta t f_x\left(t^n + \Delta t, x^n + \frac{439}{216}k_{1,x} - 8k_{2,x} + \frac{3680}{513}k_{3,x} - \frac{845}{4104}k_{4,x}, y^n + \frac{439}{216}k_{1,y} - 8k_{2,y} + \frac{3680}{513}k_{3,y} - \frac{845}{4104}k_{4,y}\right)$$
$$k_{5,y} = \Delta t f_y\left(t^n + \Delta t, x^n + \frac{439}{216}k_{1,x} - 8k_{2,x} + \frac{3680}{513}k_{3,x} - \frac{845}{4104}k_{4,x}, y^n + \frac{439}{216}k_{1,y} - 8k_{2,y} + \frac{3680}{513}k_{3,y} - \frac{845}{4104}k_{4,y}\right)$$
$$k_{6,x} = \Delta t f_x\left(t^n + \frac{1}{2}\Delta t, x^n - \frac{8}{27}k_{1,x} + 2k_{2,x} - \frac{3544}{2565}k_{3,x} + \frac{1859}{4104}k_{4,x} - \frac{11}{40}k_{5,x}, y^n - \frac{8}{27}k_{1,y} + 2k_{2,y} - \frac{3544}{2565}k_{3,y} + \frac{1859}{4104}k_{4,y} - \frac{11}{40}k_{5,y}\right)$$
$$k_{6,y} = \Delta t f_y\left(t^n + \frac{1}{2}\Delta t, x^n - \frac{8}{27}k_{1,x} + 2k_{2,x} - \frac{3544}{2565}k_{3,x} + \frac{1859}{4104}k_{4,x} - \frac{11}{40}k_{5,x}, y^n - \frac{8}{27}k_{1,y} + 2k_{2,y} - \frac{3544}{2565}k_{3,y} + \frac{1859}{4104}k_{4,y} - \frac{11}{40}k_{5,y}\right)$$
$$\overline{x^{n+1}} = x^n + \left(\frac{25k_{1,x}}{216} + \frac{1408k_{3,x}}{2565} + \frac{2197k_{4,x}}{4104} - \frac{k_{5,x}}{5}\right) \text{ fourth-order RK solution}$$
$$\overline{y^{n+1}} = y^n + \left(\frac{25k_{1,y}}{216} + \frac{1408k_{3,y}}{2565} + \frac{2197k_{4,y}}{4104} - \frac{k_{5,y}}{5}\right) \text{ fourth-order RK solution}$$
$$x^{n+1} = x^n + \left(\frac{16k_{1,x}}{135} + \frac{6656k_{3,x}}{12825} + \frac{28561k_{4,x}}{56430} - \frac{9k_{5,x}}{50} + \frac{2k_{6,x}}{55}\right) \text{ fifth-order RK solution}$$
$$y^{n+1} = y^n + \left(\frac{16k_{1,y}}{135} + \frac{6656k_{3,y}}{12825} + \frac{28561k_{4,y}}{56430} - \frac{9k_{5,y}}{50} + \frac{2k_{6,y}}{55}\right) \text{ fifth-order RK solution}$$
$$\text{error}_x = \frac{k_{1,x}}{360} - \frac{128k_{3,x}}{4275} - \frac{2197k_{4,x}}{75240} + \frac{k_{5,x}}{50} + \frac{2k_{6,x}}{55}$$